Reading Debrief

- Discuss Section 10.2 w/ your group.
- Are there questions we should address?

Section 11.3.1 Double Integrals over General Regions

Let D be a closed and bounded region and let $f(x,y)$ be continuous on D. Let R be a rectangle that contains D. We define a function F on R as follows

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{otherwise.} \end{cases}$$

The double integral of f over D is defined to be

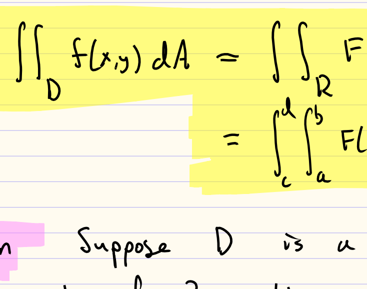
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_c^d \int_a^b F(x,y)\,dx\,dy.$$

Type I Region Suppose D is a region that lies between the graphs of 2 continuous functions of x. That is

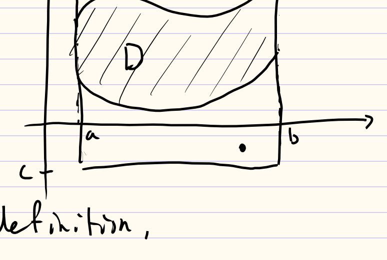
$$D = \{(x,y) : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$$

where $g_1(x), g_2(x)$ are continuous on $[a,b]$.

E.g.



Choose a rectangle $R = [a,b] \times [c,d]$ that contains D.



$c \le g_1(x) \le y \le g_2(x) \le d$

Then by definition,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx.$$

Consider $\int_c^d F(x,y)\,dy$. We have

$$\int_c^d F(x,y)\,dy = \int_{g_2(x)}^d F(x,y)\,dy + \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy + \int_c^{g_1(x)} F(x,y)\,dy$$

$$= 0 + \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy + 0.$$

Therefore, if D is a Type I Region,

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

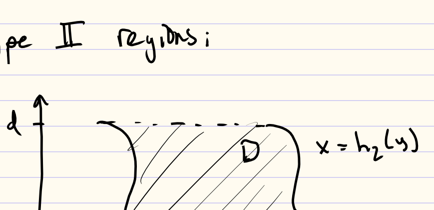
Type II Region Suppose D is a region lying between the graphs of two continuous functions of y, i.e.

$$D = \{(x,y) ; c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$$

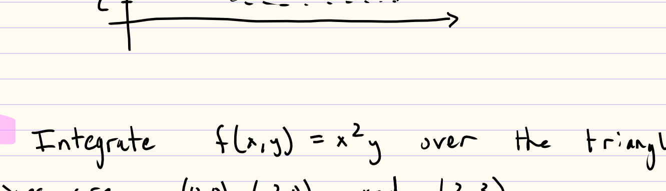
where h_1, h_2 are continuous on $[c,d]$. Then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Examples of type II regions:



Example Integrate $f(x,y) = x^2y$ over the triangle w/ vertices are $(0,0)$, $(2,0)$, and $(2,3)$.



Type II $D = \{(x,y) : 0 \le y \le 3 \text{ and } \frac{2}{3}y \le x \le 2\}$

Then

$$\iint_D x^2y\,dA = \int_0^3 \left(\int_{2/3 y}^2 x^2y\,dx\right)dy$$

$$= \int_0^3 \left[\frac{1}{3}x^3y\right]_{2/3 y}^2 dy$$

$$= \int_0^3 \frac{8}{3}y - \frac{1}{3}\cdot\frac{8}{27}y^4\,dy$$

$$= \left[\frac{8}{6}y^2 - \frac{8}{81}\cdot\frac{1}{5}y^5\right]_0^3$$

$$= \frac{8\cdot 9}{6} - 8 \cdot \frac{1}{5} \cdot 3 = ?$$

Type I $D = \{(x,y) : 0 \le x \le 2 \text{ and } 0 \le y \le \frac{3}{2}x\}$

Then

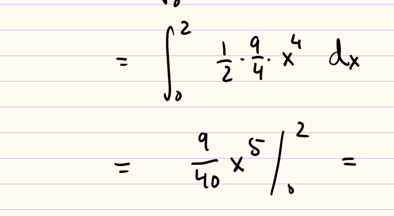
$$\iint_D f(x,y)\,dA = \int_0^2 \int_0^{3/2 x} x^2y\,dy\,dx$$

$$= \int_0^2 \left[\frac{1}{2}x^2y^2\right]_0^{3/2 x} dx$$

$$= \int_0^2 \frac{1}{2}\cdot\frac{9}{4}\cdot x^4\,dx$$

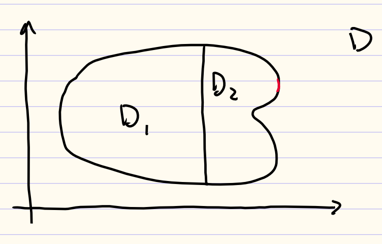
$$= \frac{9}{40}x^5 \Big|_0^2 = \frac{9}{40}\cdot 32.$$

Example A region that is not type I (but is Type II)



A nice property: Suppose $D = D_1 \cup D_2$ is the union of two regions D_1, D_2 such that D_1, D_2 do not intersect, except possibly at the boundary. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

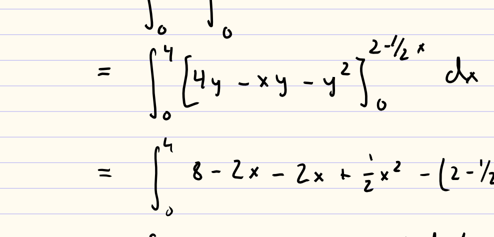


Activity 11.3.2

- Complete Activity 11.3.2 and discuss w/ your group.
- Class discussion.

Consider $\iint_D 4-x-2y\,dA$ where D is a triangle w/ vertices $(0,0)$, $(4,0)$, and $(0,2)$.

a.



$$\iint_D 4-x-2y\,dA = \int_0^4 \int_0^{2-1/2 x} 4-x-2y\,dy\,dx$$

$$= \int_0^4 \left[4y - xy - y^2\right]_0^{2-1/2 x} dx$$

$$= \int_0^4 8 - 2x - 2x + \frac{1}{2}x^2 - (2 - 1/2 x)^2\,dx$$

= for the motivated student...

$$= 16/3.$$

b.

$$\iint_D 4-x-2y\,dA = \int_0^2 \int_0^{4-2y} 4-x-2y\,dx\,dy$$

$$= \int_0^2 \left[4x - \frac{1}{2}x^2 - 2xy\right]_0^{4-2y} dy$$

$$= \int_0^2 16 - 8y - \frac{1}{2}(4-2y)^2 - 8y + 4y^2\,dy$$

$$\vdots$$

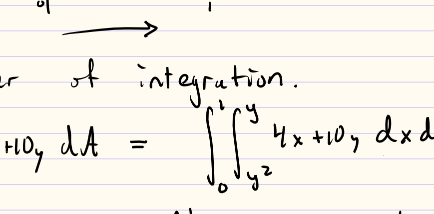
$$= 16/3.$$

Activity 11.3.3

- Complete Activity 11.3.3 and discuss w/ your group.
- Class discussion.

Given $\iint_D 4x + 10y\,dA = \int_0^1 \int_x^{\sqrt{x}} (4x+10y)\,dy\,dx$

a. Sketch the region D.



b. Switch order of integration.

$$\iint 4x+10y\,dA = \int_0^1 \int_{y^2}^y 4x+10y\,dx\,dy.$$

$$= \int_0^1 \left[2x^2 + 10xy\right]_{y^2}^y dy$$

$$= \int_0^1 2y^2 + 10y^2 - 2y^4 - 10y^3\,dy$$

$$= 4y^3 - \frac{2}{5}y^5 - \frac{10}{4}y^4 \Big|_0^1$$

$$= 4 - \frac{2}{5} - \frac{10}{4}$$

$$= \frac{80}{20} - \frac{8}{20} - \frac{50}{20} = \frac{22}{20} = \frac{11}{10}.$$

d. $A(D) = \int_0^1 \sqrt{x} - x\,dx = \frac{1}{6}.$

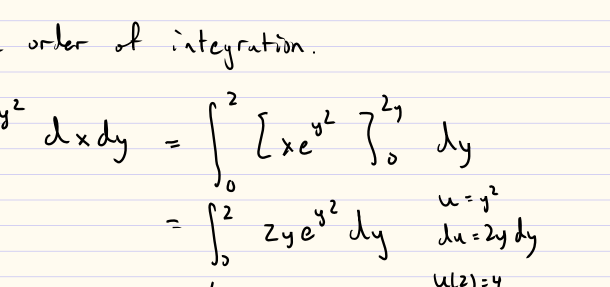
e. $f_{avg} = \dfrac{\iint_D f(x,y)\,dA}{A(D)} = \dfrac{66}{10}.$

Activity 11.3.4

- Complete Activity 11.3.4 and discuss w/ your group.
- Class discussion.

Given $\int_0^4 \int_{x/2}^2 e^{y^2}\,dy\,dx$

b. Sketch the region of integration.



c, d. Switch the order of integration.

$$\int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy = \int_0^2 \left[xe^{y^2}\right]_0^{2y} dy$$

$$= \int_0^2 2ye^{y^2}\,dy \qquad u = y^2,\ du = 2y\,dy,\ u(2) = 4,\ u(0) = 0$$

$$= \int_0^4 e^u\,du$$

$$= e^u\Big|_0^4 = e^4 - 1.$$